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Half-Differentials versus Spinor Formalism for Fermions in Low-Dimensional Systems

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The description of fermions on curved manifolds or in curvilinear coordinates usually requires a *vielbein* formalism to define Dirac γ-matrices or Pauli matrices on the manifold. Derivatives of the *vielbein* also enter equations of motion for fermions through the spin connection, which gauges local rotations or Lorentz transformations of tangent planes.

The present paper serves a dual purpose. First we will see how the *zweibein* formalism on surfaces emerges from constraining fermions to submanifolds of Minkowski space. In the second part, I will explain how in two dimensions the *zweibein* can be absorbed into the spinors to form half-order differentials. The interesting point about half-order differentials is that their derivative terms along a two-dimensional submanifold of Minkoski space look exactly like ordinary spinor derivatives in Cartesian coordinates on a planar surface, although there is a prize to pay in the form of a local mass term. The advantage of half-order differentials is that they allow for the use of the conformal field concept of conformal field theory even in low-dimensional fermion systems without conformal symmetry.

KEY WORDS: spinors; fermions in low-dimensional systems.

1. INTRODUCTION

Low-dimensional electron systems play an important role in the theoretical modeling of surfaces and interfaces in condensed matter physics, and naturally also arise in the description of quasi one-dimensional systems and quantum wires. Interfaces are crucial for thermodynamic, magnetic and conductivity properties of materials (Lüth, 2001; Mönch, 2001). The study of quasi one-dimensional systems is driven by the desire to understand the properties of particular materials with distinguished one-dimensional subsystems and also teaches us important lessons on general magnetic interactions in the more easily analyzed framework of spin chains.

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The purpose of the present note is to draw attention to the existence of a mathematical formalism for the description of low-dimensional fermions, which is equivalent to the standard spinor formalism but closely related to techniques used in two-dimensional conformal field theory. The latter usually invokes conformal symmetry of a critical system, and assumes a phenomenological description of degrees of freedom in terms of fields of given conformal weights. However, as will be pointed out below, components of spinor wave functions reduced to two-dimensional submanifolds of Minkowski space can be mapped into conformal fields of weights $(1/2, 0)$ and $(0, 1/2)$ in every low-dimensional system. This observation should be helpful in bridging the gap between the rich body of work on low-dimensional systems, which uses conventional spinor formalism, and the more specialized work in two-dimensional conformal field theory. The motivation for the present work goes back to the observation that fermionic degrees of freedom on the world sheets of superstring theory are naturally described in terms of half-order differentials (Dick, 1992). The major difference to low-dimensional systems in condensed matter physics is that the fermionic degrees of freedom of superstrings come without mass terms on the world sheet, while low-dimensonional electrons or fermionic quasi-particles have mass. Given the importance of low-dimensional systems for electronics, nanotechnology and materials science, it appears prudent to draw attention to the possibility of a complementary description of low-dimensional fermions.

Half-order differentials (or half-differentials, for short) Ψ are defined through their characteristic transformation behavior under coordinate changes. This transformation behavior is most easily expressed in terms of conformal transformations of conformal or isothermal coordinates z and $\bar{z}$ on the surface (Hawley and Schiffer, 1966),

$$z \to z'(z), \quad \bar{z} \to \bar{z}'(\bar{z}),$$

$$\Psi_{\sqrt{z}}(z, \bar{z}) \to \Psi'_{\sqrt{z}}(z', \bar{z}') = \Psi_{\sqrt{z}}(z, \bar{z})\sqrt{\frac{dz}{dz'}}, \tag{1}$$

$$\Psi_{\sqrt{\bar{z}}}(z, \bar{z}) \to \Psi'_{\sqrt{\bar{z}}}(z', \bar{z}') = \Psi_{\sqrt{\bar{z}}}(z, \bar{z})\sqrt{\frac{d\bar{z}}{d\bar{z}'}},$$

see Section 3 and the appendix for a more detailed explanation and for the definition of conformal coordinates on curved surfaces. In the language of two-dimensional conformal field theory, half-differentials are conformal fields of conformal weights $(1/2, 0)$ or $(0, 1/2)$, respectively. Two-dimensional conformal field theory is usually applied in the theory of two-dimensional critical models, where the conformal weights of the fields are part of the critical exponents of a specific model. One objective of the present paper is to point out that "conformal fields" in two dimensions also appear naturally in the description of low-dimensional systems, without being necessarily tied to critical phenomena or conformal invariance. We will proceed through most of this paper using isothermal coordinates, after introducing the proper notion of conformal gauge for coordinates in Section 3. However, I would like to emphasize that isothermal coordinates are convenient, but not necessary for the definition of half-differentials in terms of a factorized transformation law like (1) under two-dimensional coordinate transformations. The covariant generalization of Eq. (1) for arbitrary sets of coordinates on surfaces was found in Dick (1992) and is described in the appendix.

The name half-order differential seems to have been coined by Hawley and Schiffer (1966), and stems from their geometric invariance property,

$$\Psi'_{\sqrt{z}}(z', \bar{z}')\sqrt{dz'} = \Psi_{\sqrt{z}}(z, \bar{z})\sqrt{dz}, \quad \Psi'_{\sqrt{\bar{z}}}(z', \bar{z}')\sqrt{d\bar{z}'} = \Psi_{\sqrt{\bar{z}}}(z, \bar{z})\sqrt{d\bar{z}}. \tag{2}$$

In Sections 3 and 4 we will identify one-to-one mappings between low-dimensional spinors and half-differentials. The mappings are trivial in Cartesian coordinates on planar surfaces or static wires, but in general coordinate systems or on curved surfaces, or on wires with time-dependent shapes, the description of fermions in terms of half-differentials is complementary to the use of spinors. Since the connection between half-differentials and spinors is tied to two dimensions, it works best for equilibrium phenomena on surfaces or dynamical phenomena on wires. However, it is instructive to see how the mapping affects time derivatives for fermions on a surface, and therefore we will retain the time derivative terms when performing the transformation of the Lagrangian for surface electrons.

For the conventions concerning the counting of dimensions, a "three-dimensional" spinor is a spinor in four-dimensional Minkowski space, a "two-dimensional" spinor is a spinor which may depend on two space-like coordinates and time, and a "one-dimensional" spinor describes motions of a fermion on a wire. The coordinates on a space-like surface or a wire will be denoted by $\{\xi^1, \xi^2\}$ or w, respectively. In the case of fermions on a space-like surface, the shorthand notation $f(\xi, t) \equiv f(\xi^1, \xi^2, t)$ is used.

For the outline of the paper, we will first consider the reduction of three-dimensional bulk spinors to two-dimensional spinors on surfaces in Section 2. The mapping between two-dimensional spinors and half-differentials will be established in Section 3. The corresponding mapping for fermions on a wire is introduced in Section 4. Section 5 contains a brief comment on the existence of spinors and half-differentials on two-dimensional manifolds, and Section 6 explains the mapping between spinors and half-differentials in the particular case of spherical surfaces. Section 7 contains our conclusions. The generalization of Eq. (1) and the general form of the mapping between low-dimensional spinors and half-differentials for non-isothermal coordinates is given in the appendix.

2. SPINORS ON SURFACES FROM SPINORS IN MINKOWSKI SPACE

A proper derivation of the connection between spinors and half-differentials proceeds through the relativistic formulation for fermions. The setting is a static surface $\mathcal{S}$ in a flat ambient three-dimensional space. Our restricted space-time arena for particles moving on $\mathcal{S}$ is therefore $\mathcal{S} \times R$, where R stands for the time t. The ambient four-dimensional Minkowski space is triangulated with inertial coordinates $x^0 = ct$ and x^i, $1 \le i \le 3$, and local coordinates on the surface $\mathcal{S}$ are denoted by ξ^a, $1 \le a \le 2$. Unit vectors along the coordinate axes in Minkowski space are denoted by $\vec{u}_\mu$, $\vec{u}_0 \cdot \vec{u}^0 = 1$, $\vec{u}_i^2 = 1$. Greek indices μ, ν from the middle of the alphabet take values $0 \le \mu, \nu \le 3$ and refer to vectors and tensor components in an inertial basis of four-dimensional Minkowski space. Greek indices α, β, γ from the beginning of the alphabet take values $0 \le \alpha, \beta, \gamma \le 2$ and refer to a coordinate basis on the generically curved three-dimensional space $\mathcal{S} \times R$.

Embeddings of local coordinate patches Ξ of the static surface $\mathcal{S}$ in Minkowski space are given by $x^i = x^i(\xi^1, \xi^2)$. The induced tangent vectors along the coordinate lines on $\mathcal{S} \times R$ are

$$\vec{e}_a(\xi) = \partial_a \vec{x}(\xi) = \sum_{i=1}^{3} \vec{u}_i \partial_a x^i(\xi), \tag{3}$$

and the induced metric on the surface is

$$g_{ab}(\xi) = \vec{e}_a(\xi) \cdot \vec{e}_b(\xi) = \partial_a \vec{x}(\xi) \cdot \partial_b \vec{x}(\xi). \tag{4}$$

Dual basis vectors in the tangent planes of $\mathcal{S}$ are

$$\vec{e}^a(\xi) = \sum_{b=1}^{2} g^{ab}(\xi) \vec{e}_b(\xi), \quad g^{ab}(\xi) = \vec{e}^a(\xi) \cdot \vec{e}^b(\xi).$$

A projector of vectors onto the tangent space at the point ξ on $\mathcal{S}$ is

$$\underline{P}(\xi) = \sum_{a=1}^{2} \vec{e}_a(\xi) \otimes \vec{e}^a(\xi). \tag{5}$$

The Christoffel symbols on $\mathcal{S}$

$$\Gamma^a{}_{bc}(\xi) = \vec{e}^a(\xi) \cdot \partial_c \vec{e}_b(\xi), \quad \sum_{a=1}^{2} \vec{e}_a(\xi) \Gamma^a{}_{bc}(\xi) = \underline{P}(\xi) \cdot \partial_c \vec{e}_b(\xi), \tag{6}$$

define the covariant derivatives of a tangent vector

$$\vec{v}(\xi) = \sum_{a=1}^{2} v^a(\xi) \vec{e}_a(\xi)$$

through the projection of the partial derivatives onto the tangent spaces,

$$D_a\vec{v}(\xi) = \underline{P}(\xi) \cdot \partial_a\vec{v}(\xi).$$

Local coordinates in a neighbourhood $\mathcal{N}$ containing the surface coordinate patch Ξ are given by $\{\xi^1, \xi^2, \xi^\perp\}$, and we choose $\xi^\perp = 0$ on the surface (e.g. $\xi^\perp = r - R$ on a sphere of radius R). Locally the map $\{\xi^1, \xi^2, \xi^\perp\} \leftrightarrow \{x^1, x^2, x^3\}$ is an isomorphism, and the dual basis vectors on $\mathcal{S}$ can be written as

$$\vec{e}^a = \sum_{i=1}^{3} \vec{u}^i \partial_i \xi^a \Big|_{\xi^\perp=0}. \tag{7}$$

We could go from Eq. (6) straight into discussions of the Dirac equation on $\mathcal{S} \times R$, using standard *vielbein* techniques for spinors on curved manifolds. But it is more instructive to actually follow the emergence of the Dirac equation on $\mathcal{S} \times R$ from the Dirac equation in the ambient space, in a simple electron–surface interaction model. This motivates the discussion of the Dirac equation on the surface, and explains the emergence of the *zweibein* and the spin connection on the surface. We therefore assume that electrons are attracted to the surface $\mathcal{S}$ through a potential

$$V(\xi^\perp) = -e\Phi(\xi^\perp) = -W\Theta(\ell - |\xi^\perp|) \tag{8}$$

with $0 < W < 2mc^2$,

$$\gamma^0 \left(\mathrm{i}\,\hbar c\partial_0 + W\Theta(\ell - |\xi^\perp|)\right) \psi(\vec{x}, t) + \mathrm{i}\,\hbar c\boldsymbol{\gamma} \cdot \nabla\psi(\vec{x}, t) - mc^2\psi(\vec{x}, t) = 0. \tag{9}$$

We assume $W < 2mc^2$ to avoid pair creation at the potential threshold, which is the source of the Klein paradox (otherwise the potential would have to be treated as a dynamical field, which would at least partly decay due to pair creation). For electrons of energy $E < W$, the potential will imply an exponential fall off outside of the surface over a bulk penetration length $\hbar c/\sqrt{m^2c^4 - (W - E)^2}$, and eventually we can neglect bulk effects and gradients orthogonal to $\mathcal{S}$ on length scales on the surface which are large compared to the penetration length.

For the calculation of the induced Dirac operator and γ matrices on $\mathcal{S} \times R$, we note that in $\mathcal{N}$

$$\gamma^0\partial_0\psi(\vec{x}, t) + \boldsymbol{\gamma} \cdot \nabla\psi(\vec{x}, t)$$
$$= \gamma^0\partial_0\psi(\xi, \xi^\perp, t) + \sum_{a=1}^{2} \boldsymbol{\gamma} \cdot (\nabla\xi^a)\partial_a\psi(\xi, \xi^\perp, t) + \boldsymbol{\gamma} \cdot (\nabla\xi^\perp)\partial_\perp\psi(\xi, \xi^\perp, t).$$

The induced Dirac operator on $\mathcal{S} \times R$ is therefore

$$\gamma^0 \partial_0 + \sum_{a=1}^{2} \Gamma^a(\xi) \partial_a$$

with two-dimensional γ matrices

$$\Gamma^a(\xi) = \sum_{i=1}^{3} \gamma^i \partial_i \xi^a, \quad 1 \leq a \leq 2, \tag{10}$$

$$\{\Gamma^a(\xi), \Gamma^b(\xi)\} = -2 \sum_{i,j=1}^{3} \delta^{ij} \partial_i \xi^a \cdot \partial_j \xi^b = -2\vec{e}^a(\xi) \cdot \vec{e}^b(\xi) = -2g^{ab}(\xi). \tag{11}$$

Equation (10) provides us with a triplet of 4×4 γ matrices $\{\gamma^0, \Gamma^1(\xi), \Gamma^2(\xi)\}$ on the curved three-dimensional space-time $\mathcal{S} \times R$ in terms of flat γ matrices of the ambient bulk. The set $\{\gamma^0, \Gamma^1(\xi), \Gamma^2(\xi)\}$ of γ matrices must be reducible, because every irreducible representation of the three-dimensional Clifford algebra condition

$$\{\gamma^\alpha, \gamma^\beta\} = -2g^{\alpha\beta}$$

employs only 2×2 matrices, and there are exactly two equivalence classes of such matrices. The reduction is particularly easy to see with Dirac bases of flat γ matrices in four and three space-time dimensions.

The Dirac basis of γ matrices in four-dimensional Minkowski space is

$$\gamma^0 = \begin{pmatrix} \underline{1} & 0 \\ 0 & -\underline{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad 1 \leq i \leq 3. \tag{12}$$

All the entries are 2×2 matrices, and the matrices σ^i are the Pauli spin matrices. For representations of the two different equivalence classes of γ matrices in three-dimensional Minkowski space we can choose

$$\gamma_I^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_I^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma_I^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix},$$

$$\gamma_{II}^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma_{II}^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma_{II}^2 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \tag{13}$$

The reduction of the 4×4 γ matrices γ^μ, $0 \le \mu \le 2$, with respect to the three-dimensional γ matrices $\gamma^\mu_{I,II}$ is given in terms of the spinor decomposition

$$\psi_D = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \psi_{I,1} \\ \psi_{II,1} \\ \psi_{II,2} \\ \psi_{I,2} \end{pmatrix}. \tag{14}$$

This means that the pairs of fermion states which transform irreducibly under Lorentz transformations of the tangent space to the reduced space-time are the states which are related by charge conjugation

$$\psi_D \to \psi_{D,c} = \mathrm{i}\gamma_2 \psi_D^*,$$

i.e. the spin up electron mixes only with the spin up positron under Lorentz boosts of the three-dimensional tangent space.

We can write the reduction of the four-dimensional spinor representation with respect to spinor representations on the planes $x^3 =$ const. more conveniently with the rotation matrix

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathcal{M} \cdot \gamma^\mu \cdot \mathcal{M}^{-1} = \begin{pmatrix} \gamma_I^\mu & 0 \\ 0 & \gamma_{II}^\mu \end{pmatrix}, \quad 0 \le \mu \le 2, \tag{15}$$

and the orthogonal matrix γ^3, which mixes the two irreducible representations, becomes

$$\mathcal{M} \cdot \gamma^3 \cdot \mathcal{M}^{-1} = \gamma^1.$$

On the curved surface $\mathcal{S}$, each tangent plane carries the two equivalence classes of three-dimensional γ matrices (13), and the two equivalence classes of fermions correspond to the two different spin orientations with respect to the normal on the tangent plane. A normal component $A_\perp$ of the vector potential apparently couples the two equivalence classes.

On the other hand, if we would not have used the embedding of $\mathcal{S} \times R$ in the ambient four-dimensional Minkowski space and the ensuing induced Dirac operator, we would have employed a *zweibein* formalism for the metric $g_{ab}(\xi)$ to construct γ matrices on $\mathcal{S}$ in terms of flat two-dimensional γ matrices,

$$g^{ab}(\xi) = \sum_{i=1}^{2} e^a{}_i(\xi) e^{bi}(\xi), \quad \gamma^a(\xi) = \sum_{i=1}^{2} e^a{}_i(\xi)\gamma^i. \tag{16}$$

We can make the connection between the pair of induced 4×4 γ matrices $\{\Gamma^1(\xi), \Gamma^2(\xi)\}$ from Eq. (10), and the *zweibein* construction (16) on $\mathcal{S}$ by gauging away the γ^3 term in (10). This can be achieved through a rotation $\mathcal{R}$ of the tangent plane to $\mathcal{S}$ into the $(\vec{u}_1, \vec{u}_2)$-plane. The tangent plane is spanned by $\vec{e}_1$ and $\vec{e}_2$, and therefore we can construct the rotation by introducing the Cartesian vectors

$$\vec{n}_1 = \frac{\vec{e}_1}{|\vec{e}_1|},$$

$$\vec{n}_2 = \frac{\vec{e}_2 - (\vec{n}_1 \cdot \vec{e}_2)\vec{n}_1}{\sqrt{\vec{e}_2^{\,2} - (\vec{n}_1 \cdot \vec{e}_2)^2}} = \frac{\vec{e}_1^{\,2}\vec{e}_2 - (\vec{e}_1 \cdot \vec{e}_2)\vec{e}_1}{\sqrt{\vec{e}_1^{\,2}\left[\vec{e}_1^{\,2}\vec{e}_2^{\,2} - (\vec{e}_1 \cdot \vec{e}_2)^2\right]}},$$

$$\vec{n}_\perp = \vec{n}_1 \times \vec{n}_2 = \frac{\vec{e}_1 \times \vec{e}_2}{\sqrt{\vec{e}_1^{\,2}\vec{e}_2^{\,2} - (\vec{e}_1 \cdot \vec{e}_2)^2}}.$$

The rotation matrix is then given by

$$\mathcal{R}(\xi) = \begin{pmatrix} \vec{n}_1^{\,T} \\ \vec{n}_2^{\,T} \\ \vec{n}_\perp^{\,T} \end{pmatrix}. \tag{17}$$

The corresponding spinor representation

$$\mathcal{U}(\xi) = \mathcal{U}(\mathcal{R}(\xi)) \tag{18}$$

of the rotation will gauge away the γ^3 term in the γ matrices $\Gamma^a(\xi)$,

$$\gamma^a(\xi) = \mathcal{U}(\xi) \cdot \Gamma^a(\xi) \cdot \mathcal{U}^{-1}(\xi) = \sum_{i=1}^{2} e^a{}_i(\xi)\gamma^i. \tag{19}$$

The Clifford algebra property

$$\{\gamma^a(\xi), \gamma^b(\xi)\} = -2g^{ab}(\xi) \tag{20}$$

is also satisfied by the transformed γ matrices. We denote the resulting spinor components after the transformation with complex indices,

$$\mathcal{U} \cdot \psi_D \equiv \psi = \begin{pmatrix} \psi^{\sqrt{\bar{z}}} \\ \chi^{\sqrt{z}} \\ \chi^{\sqrt{\bar{z}}} \\ \psi^{\sqrt{z}} \end{pmatrix}. \tag{21}$$

The motivation for this designation will become apparent in Eq. (29) below.

Replacing ψ_D with $\mathcal{U}^{-1} \cdot \psi$ in the induced Dirac equation on $\mathcal{S} \times R$ will yield extra derivative terms $\partial_a \mathcal{U}^{-1}$, which correspond to the spin connection terms discussed below.

The Clifford algebra relations (20) and $\{\gamma^i, \gamma^j\} = -2\delta^{ij}$ imply the *zweibein* property (16).

The spinor representation of rotations of tangent planes of $\mathcal{S}$ is given in terms of the generator

$$S_{12} = \frac{\mathrm{i}}{2}\gamma_1 \cdot \gamma_2 = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}. \tag{22}$$

and the *zweibein* components (16) can be used in the standard way to convert the Christoffel symbols into a spin connection to gauge local rotations of the tangent planes.

In the present setting of $\mathcal{S} \times R$, we keep the inertial time coordinate fixed and also do not perform boosts or time-dependent rotations in the three-dimensional Minkowski spaces tangent to $\mathcal{S} \times R$. The spin connection then has only 2 independent coefficients due to $e^0{}_0 = 1$, $e^0{}_i = e^a{}_0 = 0$,

$$\Gamma^1{}_{2c} = e_a{}^1\left(\partial_c e^a{}_2 + \Gamma^a{}_{bc}e^b{}_2\right) = \vec{e}^{\,1} \cdot \partial_c \vec{e}_2 = -\Gamma_2{}^1{}_c, \tag{23}$$

and the spin connection is given by

$$\Omega_c(\xi) = \mathrm{i}\Gamma_{12c}(\xi)S_{12} = -\frac{1}{2}\Gamma_{12c}(\xi)\gamma_1 \cdot \gamma_2. \tag{24}$$

It gauges local rotations of the tangent planes of $\mathcal{S}$,

$$R(\xi) = \exp[\mathrm{i}\varphi(\xi)L_{12}] = \begin{pmatrix} \cos\varphi(\xi) & \sin\varphi(\xi) \\ -\sin\varphi(\xi) & \cos\varphi(\xi) \end{pmatrix},$$

$$U(\xi) = \exp[\mathrm{i}\varphi(\xi)S_{12}] = \exp\left[\frac{\mathrm{i}}{2}\varphi(\xi)\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}\right],$$

$$e'^a{}_i(\xi) = \sum_{j=1}^{2} R_i{}^j(\xi)e^a{}_j(\xi), \qquad \psi'(\xi, t) = U(\xi) \cdot \psi(\xi, t),$$

$$\overline{\psi}'(\xi, t) = \overline{\psi}(\xi, t) \cdot U^{-1}(\xi),$$

because the spin connection transforms according to

$$\Omega'_a(\xi) = U(\xi) \cdot \Omega_a(\xi) \cdot U^{-1}(\xi) + U(\xi) \cdot \partial_a U^{-1}(\xi).$$

The corresponding covariant derivatives are

$$\begin{aligned} D_a\psi(\xi, t) &= \partial_a\psi(\xi, t) + \Omega_a(\xi) \cdot \psi(\xi, t), \\ D_a\bar{\psi}(\xi, t) &= \partial_a\bar{\psi}(\xi, t) - \bar{\psi}(\xi, t) \cdot \Omega_a(\xi). \end{aligned} \tag{25}$$

It is well known that the spin connection on a surface does not appear in the fermion action if the derivative terms are split symmetrically between ψ and $\bar{\psi}$, see Eq. (30) below. This is due to the fact that the two-dimensional spin connection (24) anti-commutes both with γ^1 and γ^2,

$$\{\Omega_a, \gamma^i\} = 0, \quad 1 \leq i \leq 2.$$

Of course, the spin connection re-appears in the equations of motion through the derivatives of the *zweibein* if we insist on the use of spinors for the fermion wave functions. However, the *zweibein* and the spinor wave functions can be combined to form half-order differentials on $\mathcal{S}$, and this will eliminate the spin connection on $\mathcal{S}$ from the equations of motion.

3. FERMIONS AND HALF-DIFFERENTIALS ON SURFACES

It is not necessary, but very convenient for the discussion of half-differentials to choose the parameters ξ^a on the surface $\mathcal{S}$ in such a way that the induced metric (4) on $\mathcal{S}$ is conformally flat,

$$g_{ab}(\xi) = \exp[2\phi(\xi)]\,\delta_{ab}, \quad e_a{}^i(\xi) = \exp[\phi(\xi)]\,R_a{}^i(\xi), \tag{26}$$

where $R_a{}^i(\xi)$ can be an arbitrary local rotation matrix. If we start with arbitrary parameters $\xi^{(0)a}$ on the surface, the requirement to find new parameters ξ^a which satisfy the conformal gauge condition (26) amounts to two coupled second order differential equations which can always be solved. Modern proofs usually proceed by demonstrating convergence of the iterative solution of the conformal gauge conditions through Green's functions (Chern, 1955; Courant and Hilbert, 1962; Lehto, 1977). The gauge (26) is known as conformal gauge, and the corresponding parameters ξ^a are denoted as isothermal or conformal coordinates. In complex conformal coordinates

$$z = \xi^1 + \mathrm{i}\xi^2$$

the gauge conditions (26) read

$$g_{zz}(z,\bar{z}) = 0, \quad g_{z\bar{z}}(z,\bar{z}) = \frac{1}{2}\exp[2\phi(z,\bar{z})] = \frac{1}{2}\sqrt{g}.$$

We also introduce a corresponding complex notation for the non-holonomic index i of the *zweibein*, such that e.g. (note $\delta_{z\bar{z}} = 1/2$)

$$\begin{aligned}
e_z{}^z &= e_z{}^1 + \mathrm{i}e_z{}^2 = \frac{1}{2}e_1{}^1 - \frac{1}{2}\mathrm{i}e_2{}^1 + \frac{1}{2}\mathrm{i}e_1{}^2 + \frac{1}{2}e_2{}^2 = \exp(\phi - \mathrm{i}\alpha),\\
e_{\bar{z}}{}^{\bar{z}} &= \exp(\phi + \mathrm{i}\alpha), \quad e_z{}^{\bar{z}} = e_{\bar{z}}{}^z = 0,\\
e_{z\bar{z}} &= e_{\bar{z}z}{}^* = \frac{1}{2}\exp(\phi - \mathrm{i}\alpha), \quad e_{zz} = e_{\bar{z}\bar{z}} = 0,\\
e^z{}_z &= e^{\bar{z}}{}_{\bar{z}}{}^* = \exp(-\phi + \mathrm{i}\alpha), \quad e^z{}_{\bar{z}} = e^{\bar{z}}{}_z = 0.
\end{aligned} \tag{27}$$

The functions ϕ and α are time-independent for our static surface $\mathcal{S}$. The arbitrary local phase $\alpha(\xi)$ is the remnant of the local rotation matrix $R_a{}^i(\xi)$ in Eq. (26). Please keep in mind that the first index of a *zweibein* is always a coordinate index which transforms in a vector representation under coordinate transformations on the surface $\mathcal{S}$, while the second index is a tangent plane index which transforms under rotations of the tangent plane.

Under rotations of the tangent plane, the complex components of a tangent vector $\vec{v}$ transform according to

$$v'^z = \exp(-\mathrm{i}\varphi)v^z \tag{28}$$

while the components of the spinor (14) transform according to

$$\begin{aligned} \psi'^{\sqrt{\bar{z}}} &= \exp\left(\frac{\mathrm{i}}{2}\varphi\right)\psi^{\sqrt{\bar{z}}}, \quad \psi'^{\sqrt{z}} = \exp\left(-\frac{\mathrm{i}}{2}\varphi\right)\psi^{\sqrt{z}}, \\ \chi'^{\sqrt{z}} &= \exp\left(-\frac{\mathrm{i}}{2}\varphi\right)\chi^{\sqrt{z}}, \quad \chi'^{\sqrt{\bar{z}}} = \exp\left(\frac{\mathrm{i}}{2}\varphi\right)\chi^{\sqrt{\bar{z}}}. \end{aligned} \tag{29}$$

This explains our assignment of complex indices in Eq. (21). $(\psi^{\sqrt{z}})^2$ transforms like a tangent vector v^z under tangent plane rotations.

The Lagrange density for fermions with charge q on a curved space-time with metric

$$G_{\mu\nu} = E_\mu{}^m E_\nu{}^n \eta_{mn}$$

is

$$\begin{aligned} \mathcal{L} = \frac{1}{2}\sqrt{-G}E^\mu{}_m \big[&\mathrm{i}\,\hbar\left(\bar{\psi}\cdot\Omega_\mu - \partial_\mu\bar{\psi}\right)\cdot\gamma^m\cdot\psi + \mathrm{i}\,\hbar\bar{\psi}\cdot\gamma^m\cdot\left(\partial_\mu\psi + \Omega_\mu\cdot\psi\right) \\ &+ 2q\bar{\psi}\cdot\gamma^m A_\mu\cdot\psi\big] - mc\sqrt{-G}\,\bar{\psi}\psi. \end{aligned}$$

In the present case this reduces to

$$\begin{aligned} \mathcal{L} = g_{z\bar{z}}\big[&\mathrm{i}\,\hbar\left(\bar{\psi}\cdot\gamma^0\cdot\partial_0\psi - \partial_0\bar{\psi}\cdot\gamma^0\cdot\psi\right) + 2q\bar{\psi}\cdot\gamma^0 A_0\cdot\psi \\ &+ \mathrm{i}\,\hbar e^z{}_z\left(\bar{\psi}\cdot\gamma^z\cdot\partial_z\psi - \partial_z\bar{\psi}\cdot\gamma^z\cdot\psi\right) + 2qe^z{}_z\bar{\psi}\cdot\gamma^z A_z\cdot\psi \\ &+ \mathrm{i}\,\hbar e^{\bar{z}}{}_{\bar{z}}\left(\bar{\psi}\cdot\gamma^{\bar{z}}\cdot\partial_{\bar{z}}\psi - \partial_{\bar{z}}\bar{\psi}\cdot\gamma^{\bar{z}}\cdot\psi\right) + 2qe^{\bar{z}}{}_{\bar{z}}\bar{\psi}\cdot\gamma^{\bar{z}} A_{\bar{z}}\cdot\psi - 2mc\bar{\psi}\psi\big] \\ =& -2\mathrm{i}\mathcal{L}_{z\bar{z}}. \end{aligned} \tag{30}$$

The extraction of the factor $-2\mathrm{i}$ in the definition of $\mathcal{L}_{z\bar{z}}$ is due to

$$d\xi^1 d\xi^2 = \frac{\mathrm{i}}{2}dz d\bar{z},$$

so that the Lagrangian is

$$L = \int d\xi^1 d\xi^2\,\mathcal{L} = \int dz d\bar{z}\,\mathcal{L}_{z\bar{z}}.$$

The γ matrices with complex tangent space indices are

$$\gamma^z = \gamma^1 + \mathrm{i}\gamma^2 = \begin{pmatrix} 0 & \sigma_+ \\ -\sigma_+ & 0 \end{pmatrix}, \quad \gamma^{\bar{z}} = \gamma^1 - \mathrm{i}\gamma^2 = \begin{pmatrix} 0 & \sigma_- \\ -\sigma_- & 0 \end{pmatrix}. \tag{31}$$

We should amend the action (30) with the mixing term

$$\Delta\mathcal{L} = 2qg_{z\bar{z}}\bar{\psi} \cdot \gamma^3 A_\perp \cdot \psi \tag{32}$$

because $\mathcal{S} \times R$ is embedded in four-dimensional Minkowski space.

In the next step, we insert Eqs. (21, 31) and the adjoint spinor

$$\bar{\psi} = \psi^+\gamma^0 = \left(\psi^{*\sqrt{z}}, \chi^{*\sqrt{\bar{z}}}, -\chi^{*\sqrt{z}}, -\psi^{*\sqrt{\bar{z}}}\right) \tag{33}$$

into the sum of Eqs. (30) and (32) to find

$$\begin{aligned}
\mathcal{L}_{z\bar{z}} = g_{z\bar{z}}\Big[\frac{\hbar}{2}\Big(&\partial_0\psi^{*\sqrt{z}} \cdot \psi^{\sqrt{\bar{z}}} - \psi^{*\sqrt{z}} \cdot \partial_0\psi^{\sqrt{\bar{z}}} + \partial_0\chi^{*\sqrt{\bar{z}}} \cdot \chi^{\sqrt{z}} - \chi^{*\sqrt{\bar{z}}} \cdot \partial_0\chi^{\sqrt{z}} \\
&+ \partial_0\chi^{*\sqrt{z}} \cdot \chi^{\sqrt{\bar{z}}} - \chi^{*\sqrt{z}} \cdot \partial_0\chi^{\sqrt{\bar{z}}} + \partial_0\psi^{*\sqrt{\bar{z}}} \cdot \psi^{\sqrt{z}} - \psi^{*\sqrt{\bar{z}}} \cdot \partial_0\psi^{\sqrt{z}}\Big) \\
&+ \mathrm{i}qA_0\left(\psi^{*\sqrt{z}} \cdot \psi^{\sqrt{\bar{z}}} + \chi^{*\sqrt{\bar{z}}} \cdot \chi^{\sqrt{z}} + \chi^{*\sqrt{z}} \cdot \chi^{\sqrt{\bar{z}}} + \psi^{*\sqrt{\bar{z}}} \cdot \psi^{\sqrt{z}}\right) \\
&+ \mathrm{i}qA_\perp\left(\psi^{*\sqrt{z}} \cdot \chi^{\sqrt{\bar{z}}} - \chi^{*\sqrt{\bar{z}}} \cdot \psi^{\sqrt{z}} + \chi^{*\sqrt{z}} \cdot \psi^{\sqrt{\bar{z}}} - \psi^{*\sqrt{\bar{z}}} \cdot \chi^{\sqrt{z}}\right) \\
&- \mathrm{i}mc\left(\psi^{*\sqrt{z}} \cdot \psi^{\sqrt{\bar{z}}} + \chi^{*\sqrt{\bar{z}}} \cdot \chi^{\sqrt{z}} - \chi^{*\sqrt{z}} \cdot \chi^{\sqrt{\bar{z}}} - \psi^{*\sqrt{\bar{z}}} \cdot \psi^{\sqrt{z}}\right)\Big] \\
&+ e_{\bar{z}z}\Big[\hbar\left(\partial_z\psi^{*\sqrt{z}} \cdot \psi^{\sqrt{z}} + \partial_z\chi^{*\sqrt{z}} \cdot \chi^{\sqrt{z}} - \psi^{*\sqrt{z}} \cdot \partial_z\psi^{\sqrt{z}} - \chi^{*\sqrt{z}} \cdot \partial_z\chi^{\sqrt{z}}\right) \\
&+ 2\mathrm{i}qA_z\left(\psi^{*\sqrt{z}} \cdot \psi^{\sqrt{z}} + \chi^{*\sqrt{z}} \cdot \chi^{\sqrt{z}}\right)\Big] + e_{z\bar{z}}\Big[\hbar\Big(\partial_{\bar{z}}\psi^{*\sqrt{\bar{z}}} \cdot \psi^{\sqrt{\bar{z}}} \\
&+ \partial_{\bar{z}}\chi^{*\sqrt{\bar{z}}} \cdot \chi^{\sqrt{\bar{z}}} - \psi^{*\sqrt{\bar{z}}} \cdot \partial_{\bar{z}}\psi^{\sqrt{\bar{z}}} - \chi^{*\sqrt{\bar{z}}} \cdot \partial_{\bar{z}}\chi^{\sqrt{\bar{z}}}\Big) \\
&+ 2\mathrm{i}qA_{\bar{z}}\left(\psi^{*\sqrt{\bar{z}}} \cdot \psi^{\sqrt{\bar{z}}} + \chi^{*\sqrt{\bar{z}}} \cdot \chi^{\sqrt{\bar{z}}}\right)\Big].
\end{aligned} \tag{34}$$

This is an unwieldy looking equation, but we can combine the spinor and *zweibein* components into half-differentials

$$\Psi_{\sqrt{z}} = \sqrt{e_{z\bar{z}}}\psi^{\sqrt{\bar{z}}}, \quad \Psi_{\sqrt{\bar{z}}} = \sqrt{e_{\bar{z}z}}\psi^{\sqrt{z}}, \quad \Upsilon_{\sqrt{z}} = \sqrt{e_{z\bar{z}}}\chi^{\sqrt{\bar{z}}}, \quad \Upsilon_{\sqrt{\bar{z}}} = \sqrt{e_{\bar{z}z}}\chi^{\sqrt{z}}, \tag{35}$$

because the derivatives of the *zweibein* components will cancel in the alternating derivative terms in (34). Note that the spinor components are invariant under coordinate transformations, but transform under tangent plane rotations. The half-differentials, on the other hand, are invariant under tangent plane rotations and transform according to Eq. (1) under conformal gauge preserving coordinate transformations.

We also use the equation

$$g_{z\bar{z}} = 2e_{z\bar{z}}e_{\bar{z}z}.$$

The Lagrange density written in terms of the half-differentials (35),

$$\mathcal{L}_{z\bar{z}} = \mathcal{L}_{z\bar{z},\perp} + \mathcal{L}_{z\bar{z},m} + \mathcal{L}_{z\bar{z},\parallel}, \tag{36}$$

contains a single factor $\sqrt{e_{z\bar{z}}e_{\bar{z}z}}$ as a reminder of the background geometry in the orthogonal derivative and potential terms

$$\begin{aligned}\mathcal{L}_{z\bar{z},\perp} = \sqrt{e_{z\bar{z}}e_{\bar{z}z}}\big[\hbar\big(&\partial_0\Psi^*_{\sqrt{\bar{z}}}\cdot\Psi_{\sqrt{z}} - \Psi^*_{\sqrt{\bar{z}}}\cdot\partial_0\Psi_{\sqrt{z}} + \partial_0\Upsilon^*_{\sqrt{z}}\cdot\Upsilon_{\sqrt{\bar{z}}} - \Upsilon^*_{\sqrt{z}}\cdot\partial_0\Upsilon_{\sqrt{\bar{z}}}\\ &+ \partial_0\Upsilon^*_{\sqrt{\bar{z}}}\cdot\Upsilon_{\sqrt{z}} - \Upsilon^*_{\sqrt{\bar{z}}}\cdot\partial_0\Upsilon_{\sqrt{z}} + \partial_0\Psi^*_{\sqrt{z}}\cdot\Psi_{\sqrt{\bar{z}}} - \Psi^*_{\sqrt{z}}\cdot\partial_0\Psi_{\sqrt{\bar{z}}}\big)\\ &+ 2\mathrm{i}qA_0\big(\Psi^*_{\sqrt{\bar{z}}}\cdot\Psi_{\sqrt{z}} + \Upsilon^*_{\sqrt{z}}\cdot\Upsilon_{\sqrt{\bar{z}}} + \Upsilon^*_{\sqrt{\bar{z}}}\cdot\Upsilon_{\sqrt{z}} + \Psi^*_{\sqrt{z}}\cdot\Psi_{\sqrt{\bar{z}}}\big)\\ &+ 2\mathrm{i}qA_\perp\big(\Psi^*_{\sqrt{\bar{z}}}\cdot\Upsilon_{\sqrt{z}} - \Upsilon^*_{\sqrt{z}}\cdot\Psi_{\sqrt{\bar{z}}} + \Upsilon^*_{\sqrt{\bar{z}}}\cdot\Psi_{\sqrt{z}} - \Psi^*_{\sqrt{z}}\cdot\Upsilon_{\sqrt{\bar{z}}}\big)\big]\end{aligned}$$

and in the mass term

$$\mathcal{L}_{z\bar{z},m} = 2\mathrm{i}mc\sqrt{e_{z\bar{z}}e_{\bar{z}z}}\big(\Psi^*_{\sqrt{z}}\cdot\Psi_{\sqrt{\bar{z}}} - \Psi^*_{\sqrt{\bar{z}}}\cdot\Psi_{\sqrt{z}} + \Upsilon^*_{\sqrt{\bar{z}}}\cdot\Upsilon_{\sqrt{z}} - \Upsilon^*_{\sqrt{z}}\cdot\Upsilon_{\sqrt{\bar{z}}}\big),$$

but the derivative and potential terms in the surface look exactly like an action for a spinor on a flat plane,

$$\begin{aligned}\mathcal{L}_{z\bar{z},\parallel} = \;&\hbar\big(\partial_z\Psi^*_{\sqrt{\bar{z}}}\cdot\Psi_{\sqrt{\bar{z}}} + \partial_z\Upsilon^*_{\sqrt{\bar{z}}}\cdot\Upsilon_{\sqrt{\bar{z}}} - \Psi^*_{\sqrt{\bar{z}}}\cdot\partial_z\Psi_{\sqrt{\bar{z}}} - \Upsilon^*_{\sqrt{\bar{z}}}\cdot\partial_z\Upsilon_{\sqrt{\bar{z}}}\big)\\ &+ 2\mathrm{i}qA_z\big(\Psi^*_{\sqrt{\bar{z}}}\cdot\Psi_{\sqrt{\bar{z}}} + \Upsilon^*_{\sqrt{\bar{z}}}\cdot\Upsilon_{\sqrt{\bar{z}}}\big) + 2\mathrm{i}qA_{\bar{z}}\big(\Psi^*_{\sqrt{z}}\cdot\Psi_{\sqrt{z}} + \Upsilon^*_{\sqrt{z}}\cdot\Upsilon_{\sqrt{z}}\big)\\ &+ \hbar\big(\partial_{\bar{z}}\Psi^*_{\sqrt{z}}\cdot\Psi_{\sqrt{z}} + \partial_{\bar{z}}\Upsilon^*_{\sqrt{z}}\cdot\Upsilon_{\sqrt{z}} - \Psi^*_{\sqrt{z}}\cdot\partial_{\bar{z}}\Psi_{\sqrt{z}} - \Upsilon^*_{\sqrt{z}}\cdot\partial_{\bar{z}}\Upsilon_{\sqrt{z}}\big).\end{aligned}$$

Naively, in a gauge $\alpha(z,\bar{z}) = 0$, one could think of the mapping (35) as a scale transformation between spinors, but that interpretation is actually not correct. The mapping between spinors and half-differentials is a mapping between entities with different geometric transformation properties. It is important to keep this in mind, because only then does it become clear that the field $\Psi_{\sqrt{\bar{z}}}$ and its companions are fields with half-integer conformal weight in the parlance of two-dimensional conformal field theory, and that the resulting action S is invariant under coordinate transformations $z \to z'(z)$. Please also note that the mapping (35) is even relevant in the seemingly trivial case of a planar surface. It is only hidden if one uses Cartesian coordinates on the plane, but as soon as conformal coordinate transformations $z \to z'(z)$ are introduced, it again provides the link between the spinor and conformal field description of two-dimensional fermions.

The virtue of Eq. (36) is to provide an explicit connection between actual fermionic degrees of freedom on space-like surfaces and the corresponding notions used in two-dimensional conformal field theory. Eq. (36) also tells us that if we want to use conformal field theory techniques for dynamical phenomena and massive fermions on surfaces, we have to cope with the given background geometry and conformal coordinate transformations through the (1/2, 1/2) differential[2] $\sqrt{e_{z\bar{z}}e_{\bar{z}z}}$.

4. FERMIONS ON A WIRE

We now reduce the number of dimensions further by considering fermions moving in one space-like dimension - a wire. However, we treat this problem in more generality than electrons confined to a static space-like surface by allowing the form of the wire to change with time. The embedding of the wire in the ambient flat Minkowski space therefore has the form $w \to x^i(w, t)$, $1 \leq i \leq 3$, where w is a coordinate along the wire.

Now our two-dimensional manifold carrying the half-differentials is the world sheet $\mathcal{W}$ traced out by the wire as it moves through space-time. The induced metric on $\mathcal{W}$ has local components

$$g_{00} = -1 + (\partial_0\vec{x}(w, t))^2, \quad g_{0w} = \partial_0\vec{x}(w, t) \cdot \partial_w\vec{x}(w, t), \quad g_{ww} = (\partial_w\vec{x}(w, t))^2.$$

It is very convenient to change coordinates $t, w \to \tau, \sigma$ on $\mathcal{W}$ such that the conformal gauge conditions

$$g_{\tau\tau} + g_{\sigma\sigma} = 0, \quad g_{\tau\sigma} = 0$$

are satisfied. For surfaces with Minkowski signature the proof that conformal gauge can be achieved proceeds by covering the coordinate neighbourhoods on $\mathcal{W}$ with characteristics of the gauge conditions (Dick, 1989). The characteristics correspond to the new coordinate lines. Note that for a static wire only a local rescaling of w would be needed.

As in the previous case of $\mathcal{S}$, switching to conformal gauge is not necessary, because the covariant conformal field formalism described in the appendix also works on surfaces with Minkowski signature. But the equations are much nicer in conformal gauge.

We denote the remaining degree of freedom in the metric after conformal gauge fixing by $\phi(\tau, \sigma)$,

$$g_{\sigma\sigma} = -g_{\tau\tau} = \exp(2\phi).$$

The metric in the corresponding two-dimensional light cone coordinates

$$\xi^{\pm} = \sigma \pm \tau, \quad \partial_{\pm} = \frac{1}{2}(\partial_\sigma \pm \partial_\tau)$$

[2] Please keep in mind that only the first index of a *zweibein* transforms under coordinate transformations, while the second index transforms under rotations of tangent planes. This makes $\sqrt{e_{z\bar{z}}e_{\bar{z}z}}$ a (1/2, 1/2) differential under coordinate transformations.

is

$$g_{++} = g_{--} = 0, \quad g_{+-} = \frac{1}{2}\exp(2\phi) = \frac{1}{2}\sqrt{-g}.$$

This corresponds to *zweibein* components $e_\alpha{}^a$ (note $\eta_{+-} = 1/2$),

$$e_+{}^+ = \exp(\phi - u), \quad e_-{}^- = \exp(\phi + u), \quad e_+{}^- = e_-{}^+ = 0.$$

We will also quote the components with different index positions for reference in the calculation of the spinor Lagrangian on the wire,

$$e^+{}_+ = \exp(-\phi + u), \quad e^-{}_- = \exp(-\phi - u), \quad e^+{}_- = e^-{}_+ = 0,$$
$$e_{+-} = \frac{1}{2}\exp(\phi - u), \quad e_{-+} = \frac{1}{2}\exp(\phi + u), \quad e_{++} = e_{--} = 0.$$

The first index of the *zweibein* is always a world sheet index which transforms under coordinate transformations on the world sheet. The second index is a tangent plane index which transforms under Lorentz boosts of the tangent plane. Note that the set of orientation preserving coordinate transformations is restricted to

$$\xi^+ \to \xi'^+(\xi^+), \quad \xi^- \to \xi'^-(\xi^-),$$

because we exclusively work in conformal gauge. The arbitrary local parameter $u(\xi^+, \xi^-)$ is a consequence of the possibility to perform local Lorentz boosts in the tangent planes to the world sheet of the wire.

Next we consider a boost with parameter

$$u = \operatorname{artanh}(\beta) = \frac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right)$$

in the tangent plane at the point with coordinates $\xi = (\tau, \sigma)$. This transforms a tangent space vector $v^a(\xi)$ in the standard way,

$$v'^0 = \frac{v^0 - \beta v^1}{\sqrt{1-\beta^2}}, \quad v'^1 = \frac{v^1 - \beta v^0}{\sqrt{1-\beta^2}},$$

or in a light cone basis $v^\pm = v^1 \pm v^0$ in the tangent plane:

$$\begin{pmatrix} v'^+ \\ v'^- \end{pmatrix} = \begin{pmatrix} \left(\frac{1-\beta}{1+\beta}\right)^{1/2} & 0 \\ 0 & \left(\frac{1+\beta}{1-\beta}\right)^{1/2} \end{pmatrix} \begin{pmatrix} v^+ \\ v^- \end{pmatrix}. \tag{37}$$

In the light cone basis of the tangent plane, the only non-vanishing components of the boost matrix in the vector representation are

$$\Lambda^+{}_+ = \exp(-u) = \left(\frac{1-\beta}{1+\beta}\right)^{1/2}, \quad \Lambda^-{}_- = \exp(u) = \left(\frac{1+\beta}{1-\beta}\right)^{1/2}.$$

This is similar to the transformation of spinor components in the tangent plane. We use a Weyl basis of γ matrices in the tangent planes of $\mathcal{W}$,

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{38}$$

The spinor representation of the boost generator

$$S_{10} = \frac{\mathrm{i}}{2}\gamma_1\gamma_0 = \frac{\mathrm{i}}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

yields the spinor transformation law

$$\begin{aligned}\begin{pmatrix} \psi'^{\sqrt{+}} \\ \psi'^{\sqrt{-}} \end{pmatrix} &= \exp(\mathrm{i}uS_{10}) \cdot \begin{pmatrix} \psi^{\sqrt{+}} \\ \psi^{\sqrt{-}} \end{pmatrix} = \exp\left[-\frac{u}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]\begin{pmatrix} \psi^{\sqrt{+}} \\ \psi^{\sqrt{-}} \end{pmatrix} \\ &= \begin{pmatrix} \exp(-u/2) & 0 \\ 0 & \exp(u/2) \end{pmatrix}\begin{pmatrix} \psi^{\sqrt{+}} \\ \psi^{\sqrt{-}} \end{pmatrix} \\ &= \begin{pmatrix} \left(\frac{1-\beta}{1+\beta}\right)^{1/4} & 0 \\ 0 & \left(\frac{1+\beta}{1-\beta}\right)^{1/4} \end{pmatrix}\begin{pmatrix} \psi^{\sqrt{+}} \\ \psi^{\sqrt{-}} \end{pmatrix}.\end{aligned} \tag{39}$$

The fact that two-dimensional spinors transform with the square root of the vector representation of the Lorentz boost motivated our assignment of indices to the components of the two-dimensional Dirac spinor in the Weyl basis. $(\psi^{\sqrt{+}})^2$ and $(\psi^{\sqrt{-}})^2$ transform like the components of a tangent vector in a light cone basis.

To write down the fermion action on the wire, we need to write a few more quantities in light cone coordinates on the world sheet, or light cone bases for the tangent planes, respectively. The flat tangent plane gamma matrices in the light cone basis are

$$\gamma^+ = \gamma^1 + \gamma^0 = -\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \quad \gamma^- = \gamma^1 - \gamma^0 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \tag{40}$$

and the transformation of the integration measure to the light cone coordinates on the world sheet is

$$d\tau d\sigma = \frac{1}{2}d\xi^+ d\xi^-, \quad d\tau d\sigma\sqrt{-g} = d\xi^+ d\xi^- g_{+-}.$$

As in the previous case of fermions on a space-like surface, the spin connection

$$\Omega_\alpha = -\mathrm{i}\Gamma^0{}_{1\alpha}S^1{}_0 = \frac{1}{2}\Gamma^0{}_{1\alpha}\gamma^1\gamma_0,$$

anti-commutes both with γ^0 and γ^1 and will not appear in the Dirac action if we split the derivatives symmetrically between $\bar{\psi}$ and ψ.

The resulting action for spinors on the world sheet of the wire is

$$\begin{aligned} S &= \frac{1}{2}\int d\tau d\sigma\,\sqrt{-g}\left[\mathrm{i}\;\hbar e^{\alpha}{}_{a}\left(\bar{\psi}\cdot\gamma^{a}\cdot\partial_{\alpha}\psi-\partial_{\alpha}\bar{\psi}\cdot\gamma^{a}\cdot\psi\right)-2mc\bar{\psi}\cdot\psi\right] \\ &= \frac{1}{2}\int d\xi^{+}d\xi^{-}g_{+-}\left[\mathrm{i}\,\hbar e^{+}{}_{+}\left(\bar{\psi}\cdot\gamma^{+}\cdot\partial_{+}\psi-\partial_{+}\bar{\psi}\cdot\gamma^{+}\cdot\psi\right)\right. \\ &\quad \left.+\,\mathrm{i}\,\hbar e^{-}{}_{-}\left(\bar{\psi}\cdot\gamma^{-}\cdot\partial_{-}\psi-\partial_{-}\bar{\psi}\cdot\gamma^{-}\cdot\psi\right)-2mc\bar{\psi}\cdot\psi\right]. \end{aligned} \tag{41}$$

In the next step we insert the components of the two-dimensional Dirac spinor

$$\psi=\begin{pmatrix}\psi^{\sqrt{+}}\\ \psi^{\sqrt{-}}\end{pmatrix},\quad \bar{\psi}=(-\psi^{\sqrt{-},*},-\psi^{\sqrt{+},*})$$

and the γ-matrices (40),

$$\begin{aligned} S = \int d\xi^{+}d\xi^{-}&\left[\mathrm{i}\,\hbar e_{-+}\left(\psi^{\sqrt{+},*}\cdot\partial_{+}\psi^{\sqrt{+}}-\partial_{+}\psi^{\sqrt{+},*}\cdot\psi^{\sqrt{+}}\right)\right. \\ &-\mathrm{i}\,\hbar e_{+-}\left(\psi^{\sqrt{-},*}\cdot\partial_{-}\psi^{\sqrt{-}}-\partial_{-}\psi^{\sqrt{-},*}\cdot\psi^{\sqrt{-}}\right) \\ &\left.+\,2mce_{-+}e_{+-}\left(\psi^{\sqrt{-},*}\psi^{\sqrt{+}}+\psi^{\sqrt{+},*}\psi^{\sqrt{-}}\right)\right]. \end{aligned}$$

This can be rearranged as

$$\begin{aligned} S &= \int d\xi^{+}d\xi^{-}\left[\mathrm{i}\,\hbar\sqrt{e_{-+}}\psi^{\sqrt{+},*}\cdot\partial_{+}\left(\sqrt{e_{-+}}\psi^{\sqrt{+}}\right)\right. \\ &\quad -\mathrm{i}\,\hbar\partial_{+}\left(\sqrt{e_{-+}}\psi^{\sqrt{+},*}\right)\cdot\sqrt{e_{-+}}\psi^{\sqrt{+}}-\mathrm{i}\,\hbar\sqrt{e_{+-}}\psi^{\sqrt{-},*}\cdot\partial_{-}\left(\sqrt{e_{+-}}\psi^{\sqrt{-}}\right) \\ &\quad +\mathrm{i}\,\hbar\partial_{-}\left(\sqrt{e_{+-}}\psi^{\sqrt{-},*}\right)\cdot\sqrt{e_{+-}}\psi^{\sqrt{-}}\big] \\ &\quad \left.+\,2mce_{-+}e_{+-}\left(\psi^{\sqrt{-},*}\psi^{\sqrt{+}}+\psi^{\sqrt{+},*}\psi^{\sqrt{-}}\right)\right] \\ &= \int d\xi^{+}d\xi^{-}\left[\mathrm{i}\,\hbar\left(\Psi^{*}_{\sqrt{=}}\partial_{+}\Psi_{\sqrt{=}}-\partial_{+}\Psi^{*}_{\sqrt{=}}\cdot\Psi_{\sqrt{=}}-\Psi^{*}_{\sqrt{\mp}}\partial_{-}\Psi_{\sqrt{\mp}}\right.\right. \\ &\quad \left.\left.+\,\partial_{-}\Psi^{*}_{\sqrt{\mp}}\cdot\Psi_{\sqrt{\mp}}\right)+2mc\sqrt{e_{-+}e_{+-}}\left(\Psi^{*}_{\sqrt{\mp}}\Psi_{\sqrt{=}}+\Psi^{*}_{\sqrt{=}}\Psi_{\sqrt{\mp}}\right)\right]. \end{aligned} \tag{42}$$

The metric has completely disappeared in the kinetic terms, due to absorption into the half-differentials

$$\Psi_{\sqrt{=}}=\sqrt{e_{-+}}\psi^{\sqrt{+}},\quad \Psi_{\sqrt{\mp}}=\sqrt{e_{+-}}\psi^{\sqrt{-}}. \tag{43}$$

The spinors are invariant under coordinate transformations on the world sheet, and transform under Lorentz transformations in the tangent plane according to

$$\psi^{\sqrt{+}}(\xi^{+},\xi^{-})\to\psi'^{\sqrt{+}}(\xi^{+},\xi^{-})=\left(\Lambda^{+}{}_{+}\right)^{1/2}\psi^{\sqrt{+}}(\xi^{+},\xi^{-}),$$

$$\psi^{\sqrt{-}}(\xi^+,\xi^-) \to \psi'^{\sqrt{-}}(\xi^+,\xi^-) = \left(\Lambda^-{}_-\right)^{1/2}\psi^{\sqrt{-}}(\xi^+,\xi^-).$$

The half-differentials Ψ are invariant under Lorentz transformations of the tangent plane, but transform under world sheet coordinate transformations

$$\xi^+ \to \xi'^+(\xi^+), \quad \xi^- \to \xi'^-(\xi^-)$$

according to

$$\Psi_{\sqrt{-}}(\xi^+,\xi^-) \to \Psi'_{\sqrt{-}}(\xi'^+,\xi'^-) = \Psi_{\sqrt{-}}(\xi^+,\xi^-)\sqrt{\frac{\partial\xi^-}{\partial\xi'^-}},$$

$$\Psi_{\sqrt{+}}(\xi^+,\xi^-) \to \Psi'_{\sqrt{+}}(\xi'^+,\xi'^-) = \Psi_{\sqrt{+}}(\xi^+,\xi^-)\sqrt{\frac{\partial\xi^+}{\partial\xi'^+}}.$$

The action after inclusion of the gauge potentials is

$$\begin{aligned} S = \int d\xi^+ d\xi^- \big[& 2mc\sqrt{e_{-+}e_{+-}}\left(\Psi^*_{\sqrt{+}}\Psi_{\sqrt{-}} + \Psi^*_{\sqrt{-}}\Psi_{\sqrt{+}}\right) \\ & + \mathrm{i}\,\hbar\Psi^*_{\sqrt{-}}\partial_+\Psi_{\sqrt{-}} - \mathrm{i}\,\hbar\partial_+\Psi^*_{\sqrt{-}}\cdot\Psi_{\sqrt{-}} + 2q\Psi^*_{\sqrt{-}}A_+\Psi_{\sqrt{-}} \\ & - \mathrm{i}\,\hbar\Psi^*_{\sqrt{+}}\partial_-\Psi_{\sqrt{+}} + \mathrm{i}\,\hbar\partial_-\Psi^*_{\sqrt{+}}\cdot\Psi_{\sqrt{+}} - 2q\Psi^*_{\sqrt{+}}A_-\Psi_{\sqrt{+}}\big]. \end{aligned}$$

The geometry of the world sheet affects mass terms, and would also affect orthogonal vector potential terms, through the $(1/2, 1/2)$ differential $\sqrt{e_{-+}e_{+-}}$, but again the virtue of the equation is to make a connection between the conformal field formalism and fermions in low-dimensional systems.

5. EXISTENCE OF SPINORS AND HALF-DIFFERENTIALS IN TWO DIMENSIONS

Equations (28, 29) and (37, 39) illustrate the general 2-1 correspondence between vector and spinor representations of rotations and Lorentz transformations in the particular setting of two-dimensional spaces.[3]

For a general manifold $\mathcal{M}$ of dimension d, the 2-1 correspondence can cause problems with the construction of minimal (i.e. $2^{[d/2]}$-dimensional) spinor fields (Penrose and Rindler, 1984; Borel and Hirzebruch, 1959; Milnor, 1963). For every intersection $\mathcal{C}_i \cap \mathcal{C}_j \neq \emptyset$ of coordinate patches on $\mathcal{M}$ we have to assign spinor

[3] In two dimensions the correspondence takes the particularly simple form $\Lambda = U^2(\Lambda)$, because all irreducible representations of the abelian groups SO(2) and SO(1,1) are one-dimensional. Therefore both the two-dimensional vector and spinor representations have to split into one-dimensional representations, and there can be no further intertwining factors in the correspondence for the reduced representations.

transition matrices $U_{ij} = U_{ji}^{-1}$, which for every intersection $\mathcal{C}_i \cap \mathcal{C}_j \cap \mathcal{C}_k \neq \emptyset$ have to satisfy the consistency condition

$$U_{jk} U_{ki} U_{ij} = \mathbf{1}. \tag{44}$$

Since the consistency condition is fulfilled for vectors and $\Lambda = U^2(\Lambda)$, the two possibilities are

$$U_{jk} U_{ki} U_{ij} = \pm\mathbf{1}$$

and the question is whether the signs of all the spinor transition matrices can be assigned in such a way that the condition (44) is satisfied.

It is clear from the correspondences (35) and (43), that in two dimensions we will have an equivalent topological obstruction for the existence of half-differentials. For half-differentials we have to resolve the sign ambiguity of the square roots $(\partial z_i/\partial z_j)^{1/2}$ or $(\partial\xi_i^{\pm}/\partial\xi_j^{\pm})^{1/2}$ for all intersections $\mathcal{C}_i \cap \mathcal{C}_j \neq \emptyset$ in such a way that in all intersections $\mathcal{C}_i \cap \mathcal{C}_j \cap \mathcal{C}_k \neq \emptyset$ the consistency conditions

$$\sqrt{\frac{\partial z_k}{\partial z_j}}\sqrt{\frac{\partial z_i}{\partial z_k}}\sqrt{\frac{\partial z_j}{\partial z_i}} = 1 \tag{45}$$

or

$$\sqrt{\frac{\partial\xi_k^{\pm}}{\partial\xi_j^{\pm}}}\sqrt{\frac{\partial\xi_i^{\pm}}{\partial\xi_k^{\pm}}}\sqrt{\frac{\partial\xi_j^{\pm}}{\partial\xi_i^{\pm}}} = 1$$

are fulfilled. Both (44) and (45) amount to the same problem in Čech cohomology, which is concerned with the assignment of signs to intersections of coordinate patches. In their investigations of half-order differentials, Hawley and Schiffer have noticed that the second cohomology group $H^2(\mathcal{M}, Z_2)$ is trivial on orientable two-dimensional manifolds $\mathcal{M}$ (Hawley and Schiffer, 1966). In plain language, the sign ambiguities of transition functions of spinors or half-differentials can always be resolved in these cases.

The equivalent constraints (44, 45) thus imply a kind of topological spin theorem in two dimensions. If we want to consider two-dimensional field theories as local coordinate expressions of theories capable of living on two-manifolds of arbitrary topology, then we are constrained to integer and half-integer conformal weights for the fields, or equivalently to standard spinor or tensor representations. However, if our system has e.g. a simple specified topology and smoothly goes into a three-dimensional system at the boundaries (which e.g. can be described through a Hamiltonian containing a linear combination of bulk and surface terms, as introduced in Dick (2003)), anyons are a possibility. Please also note that the observations above do not exclude generalized statistics in many-particle systems, because statistics is also a consequence of quantization rules and representations of permutation groups, see e.g. Nayak and Wilzcek (1996), Baugh et al. (2001) and references there.

6. AN EXAMPLE: SPINORS AND HALF-DIFFERENTIALS ON A SPHERE

The constraints (44) or (45) are void in this case, because the sphere can be covered by only two sets of coordinates.

The coordinate singularities of polar coordinates at the poles are no particular concern for us in illustrating how the map from spinors to half-differentials works. But the conscientious reader can easily transform the results e.g. into stereographic coordinates.

In polar coordinates, the general *zweibein* on a sphere of radius r

$$e^{\vartheta}{}_1 = \frac{1}{r}\cos\alpha, \quad e^{\vartheta}{}_2 = \frac{1}{r}\sin\alpha, \quad e^{\varphi}{}_1 = -\frac{\sin\alpha}{r\sin\vartheta}, \quad e^{\varphi}{}_2 = \frac{\cos\alpha}{r\sin\vartheta}, \tag{46}$$

yields two different equivalence classes of local γ matrices through expansion in the basis (13)

$$\gamma_{\pm}^{\vartheta} = \frac{1}{r}\begin{pmatrix} 0 & \exp(\mp \mathrm{i}\alpha) \\ -\exp(\pm \mathrm{i}\alpha) & 0 \end{pmatrix},$$

$$\gamma_{\pm}^{\varphi} = \frac{1}{r\sin\vartheta}\begin{pmatrix} 0 & \mp \mathrm{i}\exp(\mp \mathrm{i}\alpha) \\ \mp \mathrm{i}\exp(\pm \mathrm{i}\alpha) & 0 \end{pmatrix}.$$

The gauge degree of freedom $\alpha \equiv \alpha(\vartheta, \varphi)$ arises from the possibility to locally rotate the *zweibein* in every tangent plane.

For comparison, the induced γ matrices on the sphere from its embedding are

$$\Gamma^{\vartheta} = \sum_{i=1}^{3}\gamma^i\partial_i\vartheta = \begin{pmatrix} 0 & \sigma^{\vartheta} \\ -\sigma^{\vartheta} & 0 \end{pmatrix}, \quad \Gamma^{\varphi} = \begin{pmatrix} 0 & \sigma^{\varphi} \\ -\sigma^{\varphi} & 0 \end{pmatrix},$$

with the Pauli matrices on the sphere

$$\sigma^{\vartheta} = \frac{1}{r}\begin{pmatrix} -\sin\vartheta & \exp(-\mathrm{i}\varphi)\cos\vartheta \\ \exp(\mathrm{i}\varphi)\cos\vartheta & \sin\vartheta \end{pmatrix},$$

$$\sigma^{\varphi} = \frac{1}{r\sin\vartheta}\begin{pmatrix} 0 & -\mathrm{i}\exp(-\mathrm{i}\varphi) \\ \mathrm{i}\exp(\mathrm{i}\varphi) & 0 \end{pmatrix}.$$

The rotation matrix (17) which locally maps the normal vector to $\vec{u}_3$ is

$$\mathcal{R}(\vartheta,\varphi) = \begin{pmatrix} \cos\vartheta\ \cos\varphi & \cos\vartheta\ \sin\varphi & -\sin\vartheta \\ -\sin\varphi & \cos\varphi & 0 \\ \sin\vartheta\ \cos\varphi & \sin\vartheta\ \sin\varphi & \cos\vartheta \end{pmatrix} = \exp(\mathrm{i}\vartheta L_2)\cdot\exp(\mathrm{i}\varphi L_3)\,,$$

with the standard so(3) generators in vector representation $(L_i)_{jk} = -\mathrm{i}\epsilon_{ijk}$.

The spin representation matrices

$$S_i = \frac{\mathrm{i}}{4}\sum_{j,k=1}^{3}\epsilon_{ijk}\gamma_j\gamma_k = \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$$

yield the corresponding spinor rotation matrix

$$\mathcal{U}(\vartheta,\varphi) = \exp(\mathrm{i}\vartheta S_2)\cdot\exp(\mathrm{i}\varphi S_3) = \begin{pmatrix} \mathcal{A}(\vartheta,\varphi) & 0 \\ 0 & \mathcal{A}(\vartheta,\varphi) \end{pmatrix}$$

with

$$\begin{aligned}\mathcal{A}(\vartheta,\varphi) &= \exp\Big(\frac{\mathrm{i}}{2}\vartheta\sigma_2\Big)\exp\Big(\frac{\mathrm{i}}{2}\varphi\sigma_3\Big)\\ &= \begin{pmatrix} \cos(\vartheta/2)\exp(\mathrm{i}\varphi/2) & \sin(\vartheta/2)\exp(-\mathrm{i}\varphi/2) \\ -\sin(\vartheta/2)\exp(\mathrm{i}\varphi/2) & \cos(\vartheta/2)\exp(-\mathrm{i}\varphi/2) \end{pmatrix}.\end{aligned}$$

We also need the inverse matrix

$$\mathcal{A}^{-1} = \begin{pmatrix} \cos(\vartheta/2)\exp(-\mathrm{i}\varphi/2) & -\sin(\vartheta/2)\exp(-\mathrm{i}\varphi/2) \\ \sin(\vartheta/2)\exp(\mathrm{i}\varphi/2) & \cos(\vartheta/2)\exp(\mathrm{i}\varphi/2) \end{pmatrix}$$

for the transformation of the induced γ matrices on the sphere. The transformed induced γ matrices on the sphere are

$$\gamma^{\vartheta} = \mathcal{U}\cdot\Gamma^{\vartheta}\cdot\mathcal{U}^{-1} = \begin{pmatrix} 0 & \mathcal{A}\cdot\sigma^{\vartheta}\cdot\mathcal{A}^{-1} \\ -\mathcal{A}\cdot\sigma^{\vartheta}\cdot\mathcal{A}^{-1} & 0 \end{pmatrix} = \frac{1}{r}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix},$$

$$\gamma^{\varphi} = \mathcal{U}\cdot\Gamma^{\varphi}\cdot\mathcal{U}^{-1} = \begin{pmatrix} 0 & \mathcal{A}\cdot\sigma^{\varphi}\cdot\mathcal{A}^{-1} \\ -\mathcal{A}\cdot\sigma^{\varphi}\cdot\mathcal{A}^{-1} & 0 \end{pmatrix} = \frac{1}{r\sin\vartheta}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}.$$

This corresponds to the gauge $\alpha = 0$ for the *zweibein* (46) on the sphere, and the reduction of the induced γ matrices in terms of the inequivalent bases of irreducible matrices is again conveniently expressed with the matrix (15),

$$\mathcal{M}\cdot\gamma^{\vartheta}\cdot\mathcal{M}^{-1} = \begin{pmatrix} \gamma_+^{\vartheta} & 0 \\ 0 & \gamma_-^{\vartheta} \end{pmatrix},\quad \mathcal{M}\cdot\gamma^{\varphi}\cdot\mathcal{M}^{-1} = \begin{pmatrix} \gamma_+^{\varphi} & 0 \\ 0 & \gamma_-^{\varphi} \end{pmatrix}.$$

For the mapping of the spinors to half-differentials on the sphere, we could use the covariantized conformal field formalism from the appendix. However, in agreement with the development in the previous sections, we will first switch to conformal gauge.

The conformal gauge conditions on the sphere

$$\sin^2\vartheta(\partial_\vartheta\xi^1)^2 + (\partial_\varphi\xi^1)^2 = \sin^2\vartheta(\partial_\vartheta\xi^2)^2 + (\partial_\varphi\xi^2)^2,$$
$$\sin^2\vartheta\,\partial_\vartheta\xi^1\cdot\partial_\vartheta\xi^2 + \partial_\varphi\xi^1\cdot\partial_\varphi\xi^2 = 0.$$

can easily be solved through

$$\xi^1 = \ln\tan(\vartheta/2), \quad \xi^2 = \varphi,$$

i.e. we have

$$z = \ln\tan(\vartheta/2) + \mathrm{i}\varphi$$

and

$$g_{z\bar{z}} = \frac{1}{2}r^2\sin^2\vartheta = 2r^2\frac{\exp(z+\bar{z})}{[1+\exp(z+\bar{z})]^2}.$$

Up to a phase, the non-vanishing components of the *zweibein* are

$$e_{z\bar{z}} = e_{\bar{z}z}{}^* = \frac{1}{2}r\sin\vartheta = \frac{r}{1+\exp(z+\bar{z})}\exp\left(\frac{z+\bar{z}}{2}\right).$$

In the basis, where the γ^3 components were gauged away in the induced γ matrices on the sphere, a spinor has components (cf. (27))

$$\psi = \begin{pmatrix} \psi^{\sqrt{\bar{z}}} \\ \chi^{\sqrt{z}} \\ \chi^{\sqrt{\bar{z}}} \\ \psi^{\sqrt{z}} \end{pmatrix} = \mathcal{U}\cdot\psi_D = \mathcal{U}\cdot\begin{pmatrix} \psi_{I,1} \\ \psi_{II,1} \\ \psi_{II,2} \\ \psi_{I,2} \end{pmatrix},$$

and e.g. two of the four resulting half-differentials on the sphere are (cf. (35))

$$\Psi_{\sqrt{z}} = \left(\frac{r}{1+\exp(z+\bar{z})}\right)^{1/2}\exp\left(\frac{z+\bar{z}}{4}\right)\psi^{\sqrt{\bar{z}}},$$
$$\Psi_{\sqrt{\bar{z}}} = \left(\frac{r}{1+\exp(z+\bar{z})}\right)^{1/2}\exp\left(\frac{z+\bar{z}}{4}\right)\psi^{\sqrt{z}}.$$

This may seem like an unusual parametrization for coordinates and fermions on the sphere, but the geometry is completely hidden in the tangential derivative and potential terms, while in the remaining terms it is reduced to the universal factor

$$\sqrt{e_{z\bar{z}}e_{\bar{z}z}} = \frac{r}{1+\exp(z+\bar{z})}\exp\left(\frac{z+\bar{z}}{2}\right).$$

7. CONCLUSION

The primary objective of the present paper was to establish the connection between the two known mathematical formalisms to describe fermions in two dimensions. Spinors can always be mapped into half-differentials through multiplication with square roots of *zweibein* components.

An examination of the mapping at the Lagrangian level revealed the specific advantages and disadvantages of half-order differentials compared to the equivalent, but much more common spinor formalism. The mapping from spinors to half-differentials eliminates geometry factors in derivatives and potential terms parallel to the two-dimensional space, and leaves only a universal geometry factor $\sqrt{e_{z\bar{z}}e_{\bar{z}z}}$ or $\sqrt{e_{-+}e_{+-}}$ for mass and orthogonal derivative terms. In particular, the mapping gauges away spin connection terms in equations of motion for low-dimensional fermions, at the expense of the local mass term and the position dependent factor in front of orthogonal derivative terms.

The use of half-order differentials is not limited to isothermal coordinates in two dimensions, but in other parametrizations utilizes anholonomic bases of tangent vectors, which would usually be avoided. Therefore half-differentials lend themselves naturally to general investigations of fermions in low-dimensional systems, because generic investigations of two-dimensional manifolds (like e.g. the general dynamics of string world sheets) usually rely on isothermal coordinates. For investigations within a given background geometry, the choice of preference between spinors or half-differentials will depend on how easy isothermal parameters can be found.

APPENDIX: THE COVARIANTIZED CONFORMAL FIELD FORMALISM IN TWO DIMENSIONS

We explain the covariantized conformal field formalism in the Euclidean framework. It works in a similar vein in the Minkowski domain, with the complex coordinates $z = x + \mathrm{i}y$ replaced by light cone coordinates $\xi^{\pm} = \sigma \pm \tau$ (Dick, 1989).

The conformal gauge conditions $g_{xx} = g_{yy}$, $g_{xy} = 0$ read in complex coordinates

$$g_{zz} = g_{\bar{z}\bar{z}}{}^{*} = 0. \tag{A1}$$

If $z, \bar{z}$ are complex conformal parameters such that the conformal gauge conditions (A1) are fulfilled, coordinate changes which preserve conformal gauge are limited to conformal transformations

$$z \to z'(z), \quad \bar{z} \to \bar{z}'(\bar{z}), \tag{A2}$$

except for possible reflections $z \to z'(\bar{z})$, which we will exclude in the following. The factorization (A2) of the two-dimensional diffeomorphism group into effectively one-dimensional transformations is necessary for the consistency of the definition of fields of conformal weight $(h, \bar{h})$ with the transformation law

$$\Psi(z, \bar{z}) \to \Psi'(z', \bar{z}') = \Psi(z, \bar{z}) \left(\frac{dz'}{dz}\right)^{-h} \left(\frac{d\bar{z}'}{d\bar{z}}\right)^{-\bar{h}}, \tag{A3}$$

see e.g. Cardy (1988); Itzykson and Drouffe (1989).

For the generalization of Eq. (A3) beyond the realm of conformal gauge fixing, assume now that z is any complex coordinate on the surface $\mathcal{S}$, not necessarily satisfying the conformal gauge conditions. The Beltrami parameters are then defined through

$$\mu_{\bar{z}}{}^{z} = \frac{g_{\bar{z}\bar{z}}}{g_{z\bar{z}} + \sqrt{g_{z\bar{z}}{}^2 - g_{zz}g_{\bar{z}\bar{z}}}} = \frac{g_{z\bar{z}} - \sqrt{g_{z\bar{z}}{}^2 - g_{zz}g_{\bar{z}\bar{z}}}}{g_{zz}} = \mu_{z}{}^{\bar{z}*},$$
$$\frac{g_{zz}}{g_{z\bar{z}}} = \frac{2\mu_z{}^{\bar{z}}}{1 + \mu_z{}^{\bar{z}}\mu_{\bar{z}}{}^{z}},$$

i.e. the metric can be written in terms of $g_{z\bar{z}}$ and the Beltrami parameters,

$$ds^2 = \frac{2g_{z\bar{z}}}{1 + \mu_z{}^{\bar{z}}\mu_{\bar{z}}{}^{z}}|dz + \mu_{\bar{z}}{}^{z}d\bar{z}|^2.$$

The Beltrami parameters satisfy $|\mu_{\bar{z}}{}^{z}| < 1$ and transform non-linearly under orientation preserving coordinate changes

$$z \to u(z, \bar{z}), \quad \partial_z u \cdot \partial_{\bar{z}}\bar{u} > \partial_z \bar{u} \cdot \partial_{\bar{z}} u, \tag{A4}$$

$$\mu_{\bar{u}}{}^{u} = \frac{\partial_{\bar{u}} z + \mu_{\bar{z}}{}^{z}\partial_{\bar{u}}\bar{z}}{\partial_u z + \mu_{\bar{z}}{}^{z}\partial_u \bar{z}} = \frac{\mu_{\bar{z}}{}^{z}\partial_z u - \partial_{\bar{z}} u}{\partial_{\bar{z}}\bar{u} - \mu_{\bar{z}}{}^{z}\partial_z \bar{u}}. \tag{A5}$$

Eq. (A5) implies in particular

$$\partial_{\bar{u}} - \mu_{\bar{u}}{}^{u}\partial_u = \left(\partial_{\bar{u}}\bar{z} - \mu_{\bar{u}}{}^{u}\partial_u \bar{z}\right)\left(\partial_{\bar{z}} - \mu_{\bar{z}}{}^{z}\partial_z\right), \tag{A6}$$

i.e. there exist derivative operators

$$\delta_z = \partial_z - \mu_z{}^{\bar{z}}\partial_{\bar{z}}, \quad \delta_{\bar{z}} = \partial_{\bar{z}} - \mu_{\bar{z}}{}^{z}\partial_z \tag{A7}$$

which transform simply with a factor under the general coordinate transformation (A4), and we have the composition law under $z, \bar{z} \to u, \bar{u} \to w, \bar{w}$

$$\partial_{\bar{z}}\bar{w} - \mu_{\bar{z}}{}^{z}\partial_{z}\bar{w} = \left(\partial_{\bar{z}}\bar{u} - \mu_{\bar{z}}{}^{z}\partial_{z}\bar{u}\right)\left(\partial_{\bar{u}}\bar{w} - \mu_{\bar{u}}{}^{u}\partial_{u}\bar{w}\right).$$

Therefore we can consistently generalize the definition (A3) to define a conformal field of weight $(h, \bar{h})$ through the transformation law $\Psi(z, \bar{z}) \to \Psi'(u, \bar{u})$ with

$$\begin{aligned}\Psi'(u, \bar{u}) &= \Psi(z, \bar{z})\left(\partial_{z}u - \mu_{z}{}^{\bar{z}}\partial_{\bar{z}}u\right)^{-h}\left(\partial_{\bar{z}}\bar{u} - \mu_{\bar{z}}{}^{z}\partial_{z}\bar{u}\right)^{-\bar{h}} \\ &= \Psi(z, \bar{z})\left(\partial_{u}z - \mu_{u}{}^{\bar{u}}\partial_{\bar{u}}z\right)^{h}\left(\partial_{\bar{u}}\bar{z} - \mu_{\bar{u}}{}^{u}\partial_{u}\bar{z}\right)^{\bar{h}}. \end{aligned} \tag{A8}$$

The 1-forms dual to the derivative operators (A7) are

$$\delta z = \frac{dz + \mu_{\bar{z}}{}^{z}d\bar{z}}{1 - \mu_{z}{}^{\bar{z}}\mu_{\bar{z}}{}^{z}} \tag{A9}$$

and its conjugate, and we have

$$dz\partial_{z} + d\bar{z}\partial_{\bar{z}} = \delta z\delta_{z} + \delta\bar{z}\delta_{\bar{z}},$$

and the factorized transformation properties

$$\delta u = \delta z\delta_{z}u, \quad \delta\bar{u} = \delta\bar{z}\delta_{\bar{z}}\bar{u}.$$

In a general coordinate frame, we can think of fields of conformal weight $(h, \bar{h})$ as invariant objects with local representations $\Psi(z, \bar{z})(\delta z)^{h}(\delta\bar{z})^{\bar{h}}$.

The relevance of Beltrami parameters in two-dimensional field theory was noticed for the first time by Baulieu and Bellon (1987). The bases (A7) and (A9) and the definition (A8) of the covariant conformal fields were introduced in Dick (1992).

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REFERENCES

Baugh, J., Finkelstein, D. R., Galiautdinov A., and Saller, H. (2001). *Journal of Mathematical Physics* **42**, 1489.

Baulieu, L. and Bellon, M. (1987). *Physics Letters B* **196**, 142.

Borel, A. and Hirzebruch, F. (1959). *American Journal of Mathematics* **81**, 315.

Cardy, J. L. (1988). In *Fields, Strings and Critical Phenomena*, E. Brezin and J. Zinn-Justin, eds. North-Holland, Amsterdam, p. 169.

Chern, S. S. (1955). *Proceedings of the American Mathematical Society* **6**, 771.

Courant, R. and Hilbert, D. (1962). *Methods of Mathematical Physics*, Vol. 2, Interscience Publishers, New York.

Dick, R. (1989). *Letters in Mathematical Physics* **18**, 67.

Dick, R. (1992). *Fortschritte der Physik* **40**, 519.
Dick, R. (2003). *International Journal of Theoretical Physics* **42**, 569.
Hawley, N. S. and Schiffer, M. (1966). *Acta Mathematica* **115**, 199.
Itzykson, C. and Drouffe, J.-M. (1989). *Statistical Field Theory*, Vol. 2, Cambridge University Press, Cambridge.
Lehto, O. (1977). In *Discrete Groups and Automorphic Functions*, W. J. Harveym, eds., Academic Press, London, p. 121.
Lüth, H. (2001). *Solid Surfaces, Interfaces and Thin Films*, 4th ed., Springer-Verlag, Berlin.
Milnor, J. (1963). *L'Enseignement Mathématique* **9**, 198.
Mönch, W. (2001). *Semiconductor Surfaces and Interfaces*, 3rd ed., Springer-Verlag, Berlin.
Nayak, C. and Wilczek, F. (1996). *Nuclear Physics B* **479**, 529.
Penrose, R. and Rindler, W. (1984). *Spinors and Space-time*, Vol. 1, Cambridge University Press, Cambridge.